B.Sc. Part III (Hons) 6^{th} Paper

GROUP THEORY

Direct product (Contd.)

Theorem Prove that $G_1 \times G_2$ is a group for the binary operation defined on them.

Proof. Here, G_1 and G_2 be two groups with composition in each group being multiplication.

$$G_1 \times G_2 = \{(g_1, g_2) : g_1 \in G_1, g_2 \in G_2\}$$

Also, the binary operation on $G_1 \times G_2$ is defined as below :-

$$(g_1, g_2)(h_1, h_2) = (g_1 h_1, g_2 h_2) : g_1, h_1 \in G_1, g_2, h_2 \in G_2$$

Now, we establish the four axioms necessary for a group.

I. Closure : Let $g_1, h_1 \in G_1$ and $g_2, h_2 \in G_2$

$\because$ G_1 is a group also, $g_1, h_1 \in G_1 \Rightarrow g_1 h_1 \in G_1$

Similarly, G_2 is also a group

$\therefore$ $g_2, h_2 \in G_2 \Rightarrow g_2 h_2 \in G_2$

Now $(g_1, g_2)(h_1, h_2) = (g_1 h_1, g_2 h_2) \in G_1 \times G_2$

So closure axiom is satisfied

II **Associative**

i.e. we show that

if $(g_1, g_2)(h_1, h_2), (k_1, k_2) \in G_1 \times G_2$ then

$$[(g_1, g_2)(h_1, h_2)](k_1, k_2) = (g_1, g_2)[(h_1, h_2)(k_1, k_2)]$$

Now,

$$\text{LHS} = [(g_1, g_2)(h_1, h_2)](k_1, k_2]$$

$$= [(g_1 h_1, g_2 h_2)](k_1, k_2)$$

$$= [(g_1 h_1)k_1, (g_2 h_2)k_2]$$

$$= [g_1(h_1 k_1), g_2(h_2 k_2)] \quad \text{Because } G_1 \text{ and } G_2 \text{ are groups.}$$

$$= (g_1, g_2)[(h_1 k_1), (h_2 k_2)]$$

$$= (g_1, g_2)[(h_1, h_2), (k_1, k_2)] = \text{RHS}$$

So, associative law is obeyed.

III **Identity** Let e_1, e_2 be the identity elements of G_1 and G_2 respectively.

Let $(g_1, g_2) \in G_1 \times G_2$ i.e. $g_1 \in G_1$, $g_2 \in G_2$

$$\therefore (e_1, e_2)(g_1, g_2) \Rightarrow (e_1 g_1, e_2 g_2) \in G_1 \times G_2$$

$$\Rightarrow (g_1, g_2) \in G_1 \times G_2$$

Again, $(g_1, g_2)\cdot(e_1, e_2) \Rightarrow (g_1 e_1, g_2 e_2) \in G_1 \times G_2$

$= (g_1, g_2)$ ~~$\in G_1 \times G_2$~~

So, $(e_1, e_2)(g_1, g_2) = (g_1, g_2)(e_1, e_2) = (g_1, g_2)$

So, (e_1, e_2) is the identity element of $G_1 \times G_2$.

<u>IV</u> <u>Inverse axiom</u> Let $(g_1, g_2) \in G_1 \times G_2$

i.e. $g_1 \in G_1$, $g_2 \in G_2 \Rightarrow g_1^{-1} \in G_1$ and $g_2^{-1} \in G_2$

$\Rightarrow (g_1^{-1}, g_2^{-1}) \in G_1 \times G_2$

Now, $(g_1, g_2)(g_1^{-1}, g_2^{-1}) = (g_1 g_1^{-1}, g_2 g_2^{-1})$

$= (e_1, e_2)$

Similarly, $(g_1^{-1}, g_2^{-1})(g_1, g_2) = (e_1, e_2)$

$\Rightarrow (g_1, g_2)(g_1^{-1}, g_2^{-1}) = (g_1^{-1}, g_2^{-1})(g_1, g_2) = (e_1, e_2)$

So Inverse axiom follows.

Hence, $G_1 \times G_2$ is a group w.r.t. to the composition defined.